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Date: April 5th, 2024

Math 12 Honours: section 5.6 Natural Logarithms and e

1. Simplify each of the following into a single exponent:

a) $\frac{2}{e^{-x}}$ $= 2e^{x}$	b) $\left(e^{x}\right)^{4}$ $= e^{4x}$	c) $e^{1-x}e^{3x}$ $= e^{1+2x}$
d) $e^{x}e^{-2x}$ $= e^{-x}$	e) $3e^{2x}\left(1-6e^{-3x}\right)$ $= 3e^{2x} - 18e^{-x}$	f) $4e^{3x-2}\left(2-3e^{2x}\right)$ $8e^{3x-2} - 12e^{5x-2}$

2. Evaluate each of the following without using a calculator

a) $e^{2\ln 4}$ $e^{\log_e 4^2} = 4^2 = \boxed{16}$	b) $\ln e^{3}$ $= 3(1) = \boxed{3}$	c) $-3\ln e$ $= -3 \cdot 1 = \boxed{-3}$
d) $e^{-4\ln 4}$ $= 4^{-4} = \boxed{\frac{1}{256}}$	e) $\ln \sqrt[4]{e^{5}}$ $= \frac{5}{4}\ln e = \boxed{\frac{5}{4}}$	f) $\ln\left(\frac{1}{e}\right)$ $= \ln e^{-1} = \boxed{-1}$
g) $\ln(1+e)$ $= \boxed{1.31326\ldots}$ calculator required	h) $e^{\ln 0}$ $\ln 0 = \varnothing$ $\boxed{\varnothing}$	i) $\ln 3 + 2\ln 4 - \ln 48$ $= \ln\left(\frac{3\cdot 4^2}{48}\right) = \ln(1) = \boxed{0}$

3. Reduct the following to lowest term

a) $e^{-2\ln 3+3\ln 2}$ $e^{\ln(3^{-2}\cdot 2^{3})} = \frac{1}{9}\cdot 8 = \boxed{\frac{8}{9}}$	b) $e^{-\ln\left(\frac{1}{e}\right)}$ $= \left(\frac{1}{e}\right)^{-1} = \boxed{e}$	c) $\ln\left(3^{-e}e^{3}\right)$ $= \ln\left(3^{-e}\right) + \ln\left(e^{3}\right)$ $= \boxed{-e\ln(3) + 3}$
d) $\ln\left(\frac{\sqrt{\pi}}{e}\right) = \boxed{\frac{1}{2}\ln\pi - 1}$	e) $\ln e^{2\ln e}$ $= 2\ln e = \boxed{2}$	f) $e^{\ln 3^{(\ln 4)}} \dashrightarrow e^{\ln(3^{\ln 4})}$ $= e^{\ln 4\cdot\ln 3} = \boxed{4.586\ldots}$

4. Solve for "x"

a) $e^{3x}=4$ $3x = \ln 4$ $x = \frac{\ln 4}{3} = \boxed{0.4621}$	b) $\ln x = 11$ $x = e^{11} = \boxed{59874.14}$	c) $\ln(3x-2)=4$ $3x-2 = e^{4} \Rightarrow \boxed{x = \frac{e^{4}+2}{3}}$
d) $\ln\left(e^{4-x}\right)=6$ $4-x=6 \Rightarrow \boxed{x=-2}$	e) $e^{5-3x}=4$ $5-3x = \ln 4$ $\boxed{x = \frac{5-\ln 4}{3}}$	f) $\ln x = \ln 11 + \ln 6 - \ln 3$ $\ln x = \ln\left(\frac{11\cdot 6}{3}\right)$ $\boxed{x=22}$
g) $\ln(\ln x)=4$ $e^{4} = \ln x \Rightarrow \boxed{x = e^{e^{4}}}$	h) $e^{e^{2x}}=4$ $\ln 4 = e^{2x}$ $2x = \ln(\ln 4) \Rightarrow \boxed{x = \frac{\ln(\ln 4)}{2}}$	i) $\frac{\ln\sqrt{x}}{2}=3$ $\frac{\frac{1}{2}\ln x}{2} = 3$ $\ln x = 12 \Rightarrow \boxed{x = e^{12}}$

j) $\ln(4x-1)=\frac{\log 10}{\log e}$ $\ln(4x-1)=\ln 10$ $4x-1=10 \Rightarrow \boxed{x=\frac{11}{4}}$	k) $\left(e^{\ln 4x}\right)^2=4$ $e^{\ln 4x}=2$ or $\cancel{-2}$ rej $\ln 4x=\ln 2$ $\boxed{x=\frac{1}{2}}$	l) $\ln x^e=1$ $x^e=e$ $\boxed{x=e^{\frac{1}{e}}}$
m) $(\ln x)^2-\frac{\log x}{\log e}+6=0$ $(\ln x)^2=\ln x-6$ $A^2-A+6=0$ No real roots $\varnothing$	n) $e^x-6e^{-x}=1$ $e^{2x}-6=e^x$ $e^{2x}-e^x-6=0$ $(e^x-3)(e^x+2)=0$ $e^x=-2$ $\cancel{x=\ln -2}$ rej. $e^x=3$ $\boxed{x=\ln 3}$	o) $(\ln x)^2+(\ln x)=2$ $A^2+A-2=0$ $(A+2)(A-1)=0$ $A=\ln x=-2$ $\quad$ $\ln x=1$ $\boxed{x=e^{-2}}$ $\quad$ $\boxed{x=e}$
p) $2\ln x=\ln(4x+5)$ $\ln x^2=\ln(4x+5)$ $x^2=4x+5$ $x^2-4x-5=0$ $(x-5)(x+1)=0$ $\boxed{x=5}$ $\quad$ $\cancel{x=-1}$ rej.	q) $e^x+4e^{-x}=5$ $e^{2x}+4=5e^x$ $e^{2x}-5e^x+4=0$ $(e^x-1)(e^x-4)=0$ $e^x=1$ $\quad$ $e^x=4$ $\boxed{x=0}$ $\quad$ $\boxed{x=\ln 4}$	r)

5. Express the following as a single logarithm:

a) $\frac{1}{3}\ln x+2\ln(6x+5)$ $=\boxed{\ln\left[\sqrt[3]{x}\,(6x+5)^2\right]}$	b) $3\ln x-\cancel{\ln(x^2-1)}+\cancel{\ln(x^2-1)}$ $=\boxed{3\ln x}$
c) $4\ln x+5\ln(2-x)-2\ln(1+x)$ $=\ln\left[\frac{x^4(2-x)^5}{(1+x)^2}\right]$	d) $\frac{2}{3}\ln x-3\ln(x^2-2x-8)$ $=\ln\left[\frac{\sqrt[3]{x^2}}{(x^2-2x-8)^3}\right]=\boxed{\ln\left[\frac{\sqrt[3]{x^2}}{(x-4)^3(x+2)^3}\right]}$

6. The relationsip between the elasped time “t”, in hours, since Jack took his first dose of medication, and the amount of medication M(t), in mg, in his bloodstream is modelled by the following function below.

$M(t)=30\times e^{-0.8t}$

I) How much medication will Jack have in his bloodstream after 3 hours?

$M(3)=30e^{-0.8\cdot 3}=\boxed{2.7215}$

II) How many hours will it take for Jack to have 1mg left in his bloodstream?

$30e^{-0.8t}=1 \Rightarrow -0.8t=\ln\left(\frac{1}{30}\right) \Rightarrow t=\frac{\ln\left(\frac{1}{30}\right)}{-0.8}=\boxed{4.25 \text{ hours}}$

7. The amount of money Dave has in his investment is given by the formula: $A = Pe^{rt}$. If He invests \$5000 at 2.5% interest, compounded continueously, how long will it take to double his investment?

$$10{,}000 = 5000e^{0.025t} \Rightarrow 0.025t = \ln 2 \Rightarrow t = \frac{\ln 2}{0.025} = \boxed{27.73 \text{ years}}$$

8. A radio-active substance has a half life of 2500 years. What is the equation for the amount of this substance after "t" years in the form of $A = Pe^{rt}$?

$$\boxed{A = Pe^{-.0002773}}$$

① solve for r

$$\frac{1}{2}A = Ae^{2500r} \Rightarrow r = \frac{\ln(\frac{1}{2})}{2500} = -0.0002773$$

9. TD bank offers an GIC that gives annual interest of 1.5% compounded monthly. What is the equivalent interest rate if the interest is compounded continuously?

$$A = P\left(1+\frac{0.015}{12}\right)^{12} = 1.015103556P \qquad 1.015103556P = Pe^{r} \Rightarrow r = \ln(1.015103556) = \boxed{0.014991}$$

annual return with monthly compound interest.

10. Each year, Jason's parents contributes \$2500 into his RESP account, then govt will match it with \$500. Suppose the RESP is invested in a fund that gives 8% return annually, compounded continuously, starting when Jason was born, how much will he have in the account when he turns 18?

$$A = \{[(3000e^{0.08} + 3000)e^{0.08} + 3000]e^{0.08} + 3000\}e^{0.08} + \cdots$$

$$A = 3000e^{19\times0.08} + 3000^{18\times0.08} + \cdots + 3000 = 3000\left[\frac{e^{19\times0.08}-1}{e^{0.08}-1}\right] = \boxed{\$128{,}671.54}$$

11. When $p = \sum_{k=1}^{6} k\ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p ? AMC 12B

$$p = \ln 1 + \ln 2^2 + \ln 3^3 + \ln 4^4 + \ln 5^5 + \ln 6^6 = \ln(2^{16}\cdot 3^9\cdot 5^5)$$

$$e^p = 2^{16}\cdot 3^9 \cdot 5^5 \Rightarrow \boxed{16}$$

12. Challenge: What is the value of $\lim_{xy\to1}\left(\frac{\ln x}{\ln y}+\frac{\ln y}{\ln x}\right) = ?$

$$xy = 1 \Rightarrow y = \frac{1}{x}$$

$$\lim_{xy\to1}\left[\frac{\ln x}{\ln 1 - \ln x} + \frac{\ln 1 - \ln x}{\ln x}\right] = \frac{\ln x}{-\ln x} + \frac{-\ln x}{\ln x} = \boxed{-2}$$

13. Evaluate: $\sum_{n=2}^{\infty}\frac{1}{n(n-1)^3}$, given Euler's beautiful result that: $1+\frac{1}{2^2}+\frac{1}{3^2}+....+\frac{1}{n^2}+.... = \frac{\pi^2}{6}$ [Math Circles]